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A FAMILY OF $\tilde{\mathrm{A}}_n$-GROUPS

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ABSTRACT

For each integer $n \geq 2$ and for each prime power q, we exhibit a group Γ which acts simply transitively on the set of vertices of the building of type $\tilde{\mathrm{A}}_n$ associated with the local field $\mathbb{F}_q((Y))$. This generalizes work in [2] and [8] (where $n = 2$) and in [1] (where $n \leq 4$).

§1. Introduction, notation

When K is a not necessarily commutative field with a discrete valuation v, there is a thick building $\Delta = \tilde{A}_n(K, v)$ of type $\tilde{\mathrm{A}}_n$ associated with K (see, e.g., [7]). The group $G = \mathrm{PGL}(n+1, K)$ acts transitively on the set $\mathcal{V}_\Delta$ of vertices of Δ, and it is natural to ask whether there is a subgroup Γ of G which acts *simply* transitively on $\mathcal{V}_\Delta$. The group G acts on Δ in a "type-rotating" way (see [1], [2]). In general, we call a group acting simply transitively and in a type-rotating way on the vertices of a thick building of type $\tilde{\mathrm{A}}_n$ an **$\tilde{A}_n$-group**. In this paper, for any integer $n \geq 2$ and for any prime power q, we exhibit $\tilde{\mathrm{A}}_n$-groups $\Gamma \leq G$ when $K = \mathbb{F}_q((Y))$.

In [1], $\tilde{\mathrm{A}}_n$-groups were characterized by means of the relatively simple presentations (called **$\tilde{A}_n$-triangle presentations**) they possess. In [1] and [2], $\tilde{\mathrm{A}}_n$-groups Γ were found for $n \leq 4$ by first finding $\tilde{\mathrm{A}}_n$-triangle presentations,

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then finding a field K such that the abstract group Γ with this presentation acts simply transitively on the vertices of $\tilde{A}_n(K,v)$. When $n=2$, Γ might act on a "nonclassical" building Δ, and there is no such K.

In the present paper, we do not use $\tilde{\mathrm{A}}_n$-triangle presentations, but instead work directly with a certain arithmetic lattice $\tilde{\Gamma}$ in $\mathrm{PGL}(n+1,K)$, and then identify a finite index subgroup Γ of $\tilde{\Gamma}$ which acts simply transitively on the vertices of $\tilde{A}_n(K,v)$. It does not appear possible to write down explicitly the $\tilde{\mathrm{A}}_n$-triangle presentation of Γ when $n \geq 5$.

It is clear from the results in [1] that for each $n \geq 2$ and each prime power q there can only be a finite number of nonarchimedean local fields K with residual field of order q such that $\tilde{A}_n(K,v)$ admits an $\tilde{\mathrm{A}}_n$-group. When $n=2$ and $q=2$ or 3, $K=\mathbb{F}_q((Y))$ and $K=\mathbb{Q}_q$ are the only possibilities [3]. By contrast to the case $K=\mathbb{F}_q((Y))$, we know only a small finite number of examples of p-adic $\tilde{\mathrm{A}}_n$-groups. All possible 2-adic and 3-adic $\tilde{\mathrm{A}}_2$-groups were found in [3], and examples of 5-adic and 7-adic $\tilde{\mathrm{A}}_2$-groups are given in [8]. The methods of [8] can be extended slightly to produce an example of a 5-adic $\tilde{\mathrm{A}}_3$-group.

Let us recall some basic facts about buildings of type $\tilde{\mathrm{A}}_n$. Let $d_{\mathcal{V}}(u,v)$ denote the natural graph distance on $\mathcal{V}_\Delta$. If we fix a vertex v_0 of Δ, and let $\Pi(v_0)=\{v\in\mathcal{V}_\Delta : d_{\mathcal{V}}(v_0,v)=1\}$ be the set of neighbours of v_0, then $\Pi(v_0)$ has a natural incidence structure: if $u,v\in\Pi(v_0)$ are distinct, we call u and v **incident** if u, v and v_0 lie on a common chamber of Δ. This makes $\Pi(v_0)$ a projective geometry, and in the case $\Delta=\tilde{A}_n(K,v)$ which concerns us here, $\Pi(v_0)$ is isomorphic to the projective geometry $\Pi(\mathbf{V})$, the flag complex of an $n+1$ dimensional vector space $\mathbf{V}$ over the residual field k of K. There is a **type map** $\tau:\mathcal{V}_\Delta\to\{0,1,\ldots,n\}$ such that each chamber of Δ contains one vertex of each type. We may assume that v_0 has type 0. The type map τ can be chosen so that the type $\tau(x_U)$ of the vertex $x_U\in\Pi(v_0)$ corresponding to $U\subset\mathbf{V}$ is just $\dim(U)$. When $\Delta=\tilde{A}_n(K,v)$, it is easy to give an explicit isomorphism $\Pi(v_0)\to\Pi(\mathbf{V})$: if v_0 is the lattice class $[L_0]$, then $\Pi(v_0)$ consists of the classes $[L]$ of lattices L satisfying $L_0\pi \subsetneq L \subsetneq L_0$, where $\pi\in K$ and $v(\pi)=1$, and we can associate to $[L]\in\Pi(v_0)$ the subspace $L/L_0\pi$ of $\mathbf{V}=L_0/L_0\pi$ (which is an $n+1$-dimensional vector space over k). For $g\in\mathrm{GL}(n+1,K)$, we set $\tau([gL_0])$ equal to the $i\in\{0,1,\ldots,n\}$ satisfying $i=-v(\det(g)) \pmod{n+1}$, and then $\tau([L])$ equals the dimension of $L/L_0\pi$ over k for each $[L]\in\Pi(v_0)$. In this paper, K equals $\mathbb{F}_q((Y))$, where q is a prime power and Y is an indeterminate. We shall always use $L_0=\mathbb{F}_q[[Y]]^{n+1}\subset\mathbb{F}_q((Y))^{n+1}$ as our base lattice. Recall that the stabilizer of L_0 in $\mathrm{GL}(n+1,\mathbb{F}_q((Y)))$ is $\mathrm{GL}(n+1,\mathbb{F}_q[[Y]])$.

Certain cyclic simple algebras (see, e.g., [4, Chapter 7]) play a central role in this paper, as they did in [1] and [2]. We start with the indeterminate Y. It will be convenient to also use $Z = 1 + Y$. Let $d = n + 1$, and let φ denote a generator of the Galois group of $\mathbb{F}_{q^d}$ over $\mathbb{F}_q$. Thus there is an integer k such that g.c.d.$(k, d) = 1$ and $\varphi(x) = x^{q^k}$ for each $x \in \mathbb{F}_{q^d}$. Note that φ extends naturally to an automorphism of $\mathbb{F}_{q^d}(Y) = \mathbb{F}_{q^d}(Z)$ over $\mathbb{F}_q(Y) = \mathbb{F}_q(Z)$. We form the cyclic simple algebra $\mathcal{A} = \mathbb{F}_{q^d}(Y)[\sigma]$, whose elements may be written uniquely

$$x_0 + x_1\sigma + \cdots + x_{d-1}\sigma^{d-1}, \quad \text{where } x_0, \ldots, x_{d-1} \in \mathbb{F}_{q^d}(Y), \tag{1.1}$$

and where multiplication is determined by the rules $\sigma^d = Z$ and $\sigma x \sigma^{-1} = \varphi(x)$ for $x \in \mathbb{F}_{q^d}(Y)$. Now $\mathbb{F}_{q^d}(Y)[\sigma]$ is actually a division algebra, because Z^d is the lowest power of Z which is a norm $N_{\mathbb{F}_{q^d}(Y)/\mathbb{F}_q(Y)}(\xi)$ of some $\xi \in \mathbb{F}_{q^d}(Y)$ (see, for example, [5, p. 84]), though this fact is not used below. Now consider an indeterminate X, and choose any $\beta \in \mathbb{F}_{q^d}$ such that $\mathrm{Tr}_{\mathbb{F}_{q^d}/\mathbb{F}_q}(\beta) = t_1 \neq 0$. Then the norm $N_{\mathbb{F}_{q^d}(X)/\mathbb{F}_q(X)}(1+\beta X)$ equals $1+t_1X+t_2X^2+\cdots+t_dX^d$ for certain other elements $t_2, \ldots, t_d$ of $\mathbb{F}_q$. We embed $\mathbb{F}_q(Y)$ into $\mathbb{F}_q(X)$ and $\mathbb{F}_{q^d}(Y)$ into $\mathbb{F}_{q^d}(X)$ by mapping Y to $t_1X + t_2X^2 + \cdots + t_dX^d$. The algebra $\mathbb{F}_{q^d}(X)[\sigma] \cong \mathcal{A} \otimes \mathbb{F}_q(X)$, whose elements are as in (1.1), but with the x_j's now in $\mathbb{F}_{q^d}(X)$, splits, i.e., is isomorphic to the algebra $M_{d\times d}(\mathbb{F}_q(X))$ of $d \times d$ matrices over $\mathbb{F}_q(X)$, because Z is now a norm. To give an explicit isomorphism, we modify slightly [1, Section 3] and [2, Section 4], and first define an embedding $\Psi : \mathbb{F}_{q^d}(X)[\sigma] \to M_{d\times d}(\mathbb{F}_{q^d}(X))$ via

$$x \mapsto \Psi(x) = \begin{pmatrix} x & 0 & \ldots & 0 \\ 0 & \varphi(x) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \varphi^{d-1}(x) \end{pmatrix} \qquad \text{for } x \in \mathbb{F}_{q^d}(X),$$

and

$$\sigma \mapsto \Psi(\sigma) = S = \begin{pmatrix} 0 & 1+\beta X & 0 & \ldots & 0 \\ 0 & 0 & 1+\varphi(\beta)X & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1+\varphi^{d-2}(\beta)X \\ 1+\varphi^{d-1}(\beta)X & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

Let $\xi_0, \ldots, \xi_{d-1}$ be a basis for $\mathbb{F}_{q^d}$ over $\mathbb{F}_q$, and let $Q \in \mathrm{GL}(d, \mathbb{F}_{q^d})$ have (i,j)-th entry $\varphi^j(\xi_i)$ for $i, j = 0, \ldots, d-1$. The conjugation $M \mapsto QMQ^{-1}$ maps the image of Ψ into $M_{d\times d}(\mathbb{F}_q(X))$, because the (i,j)-th entry of Q^{-1} is $\varphi^i(\eta_j)$, where $\mathrm{Tr}(\xi_i\eta_j) = \delta_{i,j}$. The map $\xi \mapsto Q\Psi(\xi)Q^{-1}$ is an isomorphism $\mathbb{F}_{q^d}(X)[\sigma] \to M_{d\times d}(\mathbb{F}_q(X))$.

There is a natural action of $\mathrm{Aut}(\mathcal{A})$ on Δ. For we can embed $\mathcal{A}$ into $M_{d\times d}(\mathbb{F}_q((Y)))$ by means of the map

$$(1.2)\quad \mathcal{A} \hookrightarrow \mathcal{A}\otimes \mathbb{F}_q(X) \cong M_{d\times d}(\mathbb{F}_q(X)) \hookrightarrow M_{d\times d}(\mathbb{F}_q((X))) \cong M_{d\times d}(\mathbb{F}_q((Y))),$$

where the last isomorphism comes from the fact that $Y \mapsto t_1X + \cdots + t_dX^d$ leads to an isomorphism $\mathbb{F}_q((Y)) \to \mathbb{F}_q((X))$. Now (1.2) embeds the group $\mathcal{A}^\times$ of invertible elements of $\mathcal{A}$ into $\mathrm{GL}(d, \mathbb{F}_q((Y)))$. This gives an embedding $\mathcal{A}^\times/Z(\mathcal{A}^\times) \to \mathrm{PGL}(d, \mathbb{F}_q((Y)))$. Finally, $\mathrm{Aut}(\mathcal{A}) \cong \mathcal{A}^\times/Z(\mathcal{A}^\times)$ by the Skolem–Noether Theorem (see [4, p. 263]), which states that each $\alpha \in \mathrm{Aut}(\mathcal{A})$ is of the form $x \mapsto axa^{-1}$ for some $a \in \mathcal{A}^\times$. Thus we get an embedding of $\mathrm{Aut}(\mathcal{A})$ into $\mathrm{PGL}(d, \mathbb{F}_q((Y)))$, which acts on Δ in the natural way.

§2. A subgroup of Aut($\mathcal{A}$) acting simply transitively

As above, fix a basis $\xi_0, \dots, \xi_{d-1}$ for $\mathbb{F}_{q^d}$ over $\mathbb{F}_q$. Then

$$\xi_0, \dots, \xi_{d-1}, \xi_0\sigma, \dots, \xi_{d-1}\sigma, \dots, \xi_0\sigma^{d-1}, \dots, \xi_{d-1}\sigma^{d-1}$$

is an ordered basis of $\mathcal{A}$ over $\mathbb{F}_q(Y)$.

Definition: Let $\tilde{\Gamma}$ denote the set of automorphisms α of $\mathcal{A}$ whose matrix with respect to this basis has entries in

$$\mathbb{F}_q\left[\frac{1}{Y}\right] = \mathbb{F}_q\left[\frac{1}{Z-1}\right].$$

Let Γ denote the set of $\alpha \in \tilde{\Gamma}$ such that for each $i, j = 0, \dots, d-1$, $\alpha(\xi_i\sigma^j) = \xi_i\sigma^j + \sum_{k=0}^{j-1} t_k\sigma^k \pmod{1/Y}$, where the t_k's are in $\mathbb{F}_{q^d}$.

If $\alpha \in \tilde{\Gamma}$, then $\alpha \in \Gamma$ means that its matrix with respect to the basis has the form

$$\begin{pmatrix} I & A_{1,2} & A_{1,3} & \cdots & A_{1,d} \\ 0 & I & A_{2,3} & \cdots & A_{2,d} \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & I & A_{d-1,d} \\ 0 & 0 & \cdots & 0 & I \end{pmatrix} \quad \text{modulo } \frac{1}{Y},$$

where the I's are $d\times d$ identity matrices, and the $A_{i,j}$ are $d \times d$ matrices with entries in $\mathbb{F}_q$.

Note that $\tilde{\Gamma}$ is a subgroup of $\mathrm{Aut}(\mathcal{A})$. For if $\alpha \in \mathrm{Aut}(\mathcal{A})$, then, regarded as a $\mathbb{F}_q(Y)$-linear map, the determinant of α is 1. For there is an $a \in \mathcal{A}^\times$ such that $\alpha(x) = axa^{-1}$ for all x. Let $A \in \mathrm{GL}(d, \mathbb{F}_q((Y)))$ correspond to a under the mapping (1.2). Then α extends to the composition of the maps $M \mapsto AM$ and $M \mapsto MA^{-1}$ on $M_{d\times d}(\mathbb{F}_q((Y)))$, and these maps have determinant $\det(A)^d$ and $\det(A)^{-d}$, respectively. It is routine to check that Γ is a subgroup of $\tilde{\Gamma}$.

The following element of $\mathcal{A}$ will be important for us:

$$b_1 = \sum_{j=0}^{d-1} \sigma^{-j} = 1 + \frac{1}{Z}\sum_{j=1}^{d-1}\sigma^j.$$

Calculating $b_1(1-\sigma^{-1})$, we easily see that

$$b_1 = \frac{Z-1}{Z}\left(1-\sigma^{-1}\right)^{-1}.$$

When $u \in \mathbb{F}_{q^d}^\times$, we denote by b_u the conjugate ub_1u^{-1} of b_1 by u.

LEMMA 2.1: *For each $u \in \mathbb{F}_{q^d}^\times$, conjugation by b_u is an element of Γ. The norm of b_u in $\mathcal{A}$ is $((Z-1)/Z)^{d-1}$.*

Proof: We must calculate $b_u\xi_i\sigma^j b_u^{-1}$. For $\xi \in \mathbb{F}_{q^d}$,

$$\begin{aligned} b_1\xi b_1^{-1} &= \Big(\sum_{k=0}^{d-1}\sigma^{-k}\Big)\xi\frac{Z}{Z-1}\left(1-\sigma^{-1}\right) \\ &= \xi + \frac{1}{Z-1}\Big\{\big(\xi-\varphi(\xi)\big) + (\varphi^{-1}(\xi)-\xi)\sigma^{d-1} + \cdots \\ &\qquad + (\varphi^{-(d-1)}(\xi) - \varphi^{-(d-2)}(\xi))\sigma\Big\}. \end{aligned}$$

Using the fact that σ and b_1 commute, we therefore have

$$\begin{aligned} b_u\xi\sigma^j b_u^{-1} &= u\big(b_1u^{-1}\xi\varphi^j(u)b_1^{-1}\big)\sigma^j u^{-1} \\ &= u\Big(u^{-1}\xi\varphi^j(u) + \frac{1}{Z-1}\sum_{k=0}^{d-1}t_k\sigma^k\Big)\sigma^j u^{-1} \\ &= \xi\sigma^j + \frac{1}{Z-1}\sum_{k=0}^{d-1}ut_k\varphi^{k+j}(u^{-1})\sigma^{k+j} \end{aligned}$$

for certain $t_k \in \mathbb{F}_{q^d}$. Thus for $j \le \ell \le d-1$ we get the term

$$\frac{1}{Z-1}ut_{\ell-j}\varphi^\ell(u^{-1})\sigma^\ell \equiv 0 \pmod{1/Y}$$

on the right, while for $0 \le \ell \le j-1$ we get the term

$$\frac{Z}{Z-1}ut_{d+\ell-j}\varphi^\ell(u^{-1})\sigma^\ell \equiv ut_{d+\ell-j}\varphi^\ell(u^{-1})\sigma^\ell \pmod{1/Y}.$$

It follows that conjugation by b_u is in Γ.

To calculate the norm $N(b_1)$ of b_1, let g_1 denote the image of b_1 in $\mathrm{GL}(d, \mathbb{F}_q((Y)))$ under the embedding (1.2). Then g_1^{-1} is a conjugate of $\frac{Z}{Z-1}(I - S^{-1})$. It is easy to calculate that $\det(I - S^{-1}) = 1 - 1/Z$, and so $N(b_1) = \det(g_1) = ((Z-1)/Z)^{d-1}$.

LEMMA 2.2: *For $u \in \mathbb{F}_{q^d}^\times$, let $g_u \in \mathrm{GL}(d, \mathbb{F}_q((Y)))$ be the image of $b_u = ub_1u^{-1} \in \mathcal{A}$ under the embedding (1.2). Let v_0 be the standard base vertex of the $\tilde{A}_n$ building Δ associated with $\mathbb{F}_q((Y))$. Then the type 1 neighbours of v_0 in Δ are precisely the vertices $g_u v_0$, $u \in \mathbb{F}_{q^d}^\times$, and $g_u v_0 = g_{u'} v_0$ if and only if $u = u'$ modulo $\mathbb{F}_q^\times$.*

Proof: The vertex v_0 is the equivalence class $[L_0]$ of the lattice $L_0 = \mathbb{F}_q[[Y]]^d \subset \mathbb{F}_q((Y))^d$. As $b_1 = Z^{-1}(Z + \sigma + \cdots + \sigma^{d-1})$, and the image of σ is QSQ^{-1}, which has entries in $\mathbb{F}_q[X]$, we see that g_1 has entries in $Z^{-1}\mathbb{F}_q[X] \subset \mathbb{F}_q[[Y]]$, and so $g_1 L_0 \subset L_0$. Also, $(Z-1)b_1^{-1} = Z(1 - \sigma^{-1}) = Z - \sigma^{d-1}$. So $Yg_1^{-1} = (Z-1)g_1^{-1}$ has entries in $\mathbb{F}_q[X] \subset \mathbb{F}_q[[Y]]$. Hence $Yg_1^{-1}L_0 \subset L_0$. Thus $YL_0 \subset g_1 L_0 \subset L_0$. Moreover, $\det(g_u) = \det(g_1) = N(b_1) = ((Z-1)/Z)^{d-1}$, the valuation of which is $d - 1$. So the vertex $g_u v_0 = [g_u L_0]$ is a neighbour of v_0 of type 1.

Suppose that $g_u v_0 = g_1 v_0$. Then $g_1^{-1} g_u$ must be a scalar multiple of an element of $\mathrm{GL}(d, \mathbb{F}_q[[Y]])$. But $g_1^{-1}g_u$ has determinant 1, which has valuation 0. Thus $g_1^{-1} g_u \in \mathrm{GL}(d, \mathbb{F}_q[[Y]])$. But $g_1^{-1} g_u$ is the image of

$$\begin{aligned} b_1^{-1} u b_1 u^{-1} &= \frac{Z}{Z-1}(1 - \sigma^{-1})u\big(1 + \sigma^{-1} + \cdots + \sigma^{-(d-1)}\big)u^{-1} \\ &= \frac{Z}{Z-1}(u - \varphi^{-1}(u)\sigma^{-1})\big(1 + \sigma^{-1} + \cdots + \sigma^{-(d-1)}\big)u^{-1} \\ &= \varphi^{-1}(u)/u + \big(u - \varphi^{-1}(u)\big)\frac{Z}{Z-1} b_1 u^{-1}. \end{aligned}$$

The image in $\mathrm{GL}(d, \mathbb{F}_q((Y)))$ of

$$\frac{Z}{Z-1} b_1 = \frac{1}{Z-1}\big(Z + \sigma + \cdots + \sigma^{d-1}\big)$$

is the conjugate by Q of $\frac{1}{Z-1}(ZI + S + \cdots + S^{d-1})$, which has diagonal entries $Z/(Z-1) \notin \mathbb{F}_q[[Z-1]] = \mathbb{F}_q[[Y]]$. Thus $g_1^{-1} g_u$ cannot be in $\mathrm{GL}(d, \mathbb{F}_q[[Y]])$ unless $u - \varphi^{-1}(u) = 0$; that is, unless $u \in \mathbb{F}_q$. The result is now clear.

LEMMA 2.3: *The group Γ acts transitively on the set $\mathcal{V}_\Delta$ of vertices of Δ.*

Proof: Lemma 2.2 shows that each vertex x of Δ satisfying $d(x, v_0) = 1$ and $\tau(x) = 1$ can be written $x = gv_0$, where $g \in \mathrm{PGL}(d, \mathbb{F}_q((Y)))$ is in the image of Γ. We next show by induction on i that the same is true for each vertex x satisfying $d(x,v_0)=1$ and $\tau(x)=i$. Assuming this for $i-1$ in place of i, and given x satisfying $d(x,v_0)=1$ and $\tau(x)=i$, we can choose a vertex x' satisfying $d(x',v_0)=1$, $d(x',x)=1$ and $\tau(x')=i-1$. We can write $x'=g'v_0$, where $g'\in \mathrm{PGL}(d,\mathbb{F}_q((Y)))$ is in the image of Γ. Then $d((g')^{-1}x,v_0)=d(x,g'v_0)=1$ and $\tau((g')^{-1}x)=1$, and so $(g')^{-1}x=g_uv_0$ for some $u\in\mathbb{F}_{q^d}^{\times}$. Thus $x=g'g_uv_0$ is of the desired form.

It is then easy to see by induction on $d(x,v_0)$ that each vertex x of Δ can be written $x=gv_0$, where $g\in\mathrm{PGL}(d,\mathbb{F}_q((Y)))$ is in the image of Γ. This proves the lemma.

We next describe the stabilizer of the vertex v_0 in $\mathrm{Aut}(\mathcal{A})$ and in $\tilde{\Gamma}$.

LEMMA 2.4: *Let $\alpha\in\mathrm{Aut}(\mathcal{A})$. Then α fixes the vertex v_0 if and only if it is conjugation by an element $a\in\mathcal{A}^{\times}$ which can be chosen so that $a=a_0+\cdots+a_{d-1}\sigma^{d-1}$ and $a^{-1}=a'_0+\cdots+a'_{d-1}\sigma^{d-1}$, where all the a_j's and a'_j's are in $\mathbb{F}_{q^d}[[Y]]$ (as well as in $\mathbb{F}_{q^d}(Y)$).*

Proof: Pick any $a\in\mathcal{A}^{\times}$ such that α is conjugation by a. Suppose that $A\in \mathrm{GL}(d,\mathbb{F}_q((Y)))$ corresponds to a under the embedding (1.2). By assumption, α fixes v_0. This means that $[AL_0]=[L_0]$, and so $AL_0=Y^kL_0$ for some $k\in\mathbb{Z}$. Replacing a by $Y^{-k}a$, we can assume that $AL_0=L_0$. Thus $A\in\mathrm{GL}(d,\mathbb{F}_q[[Y]])$. Write $a=a_0+a_1\sigma+\cdots+a_{d-1}\sigma^{d-1}$. Then $A=Q(A_0+A_1S+\cdots+A_{d-1}S^{d-1})Q^{-1}$, where $A_j=\Psi(a_j)$ and $S=\Psi(\sigma)$, in the notation of Section 1. As $Q\in\mathrm{GL}(d,\mathbb{F}_{q^d})$, the entries of $A_0+A_1S+\cdots+A_{d-1}S^{d-1}$ must be in $\mathbb{F}_{q^d}[[Y]]$. Now $S,\dots,S^{d-1}$ have diagonal entries all 0. So A_0 has diagonal entries in $\mathbb{F}_{q^d}[[Y]]$. In particular, the $(1,1)$ entry, a_0, is in $\mathbb{F}_{q^d}[[Y]]$. Using the fact that $S\in\mathrm{GL}(d,\mathbb{F}_{q^d}[[Y]])$, we can easily see in a similar way that $a_1,\dots,a_{d-1}$ are all in $\mathbb{F}_{q^d}[[Y]]$. Writing $a^{-1}=a'_0+\cdots+a'_{d-1}\sigma^{d-1}$, we have $A^{-1}=Q(A'_0+A'_1S+\cdots+A'_{d-1}S^{d-1})Q^{-1}$, where $A'_j=\Psi(a'_j)$, and we find that $a'_0,\dots,a'_{d-1}$ are also all in $\mathbb{F}_{q^d}[[Y]]$ for the same reasons.

Conversely, if all the a_j's and a'_j's are in $\mathbb{F}_{q^d}[[Y]]$, then it is clear that both A and A^{-1} have entries in $\mathbb{F}_q[[Y]]$, and so α fixes v_0.

LEMMA 2.5: *Let $\alpha\in\tilde{\Gamma}$. Then α fixes the vertex v_0 if and only if α is conjugation by an element of the form $\theta\sigma^j$, where $\theta\in\mathbb{F}_{q^d}^{\times}$.*

Proof: It is clear that if α is conjugation by $\theta\sigma^j$, where $\theta\in\mathbb{F}_{q^d}^{\times}$, then $\alpha\in\tilde{\Gamma}$ and α fixes v_0. Conversely, suppose that $\alpha\in\tilde{\Gamma}$ fixes v_0. Then α is conjugation by an element $a\in\mathcal{A}^{\times}$ as in Lemma 2.4. If a_r is the first of the coefficients a_j which is nonzero, then composing α with conjugation by σ^{-r}, we may suppose that

$r=0$. In addition, for each $i,j=0,\dots,d-1$, $a\xi_i\sigma^j a^{-1}$ must be a $\mathbb{F}_q[1/Y]$-linear combination of the $\xi_k\sigma^\ell$'s, and so an $\mathbb{F}_{q^d}[1/Y]$-linear combination of the σ^ℓ's. To find the coefficient of σ^ℓ in

$$\begin{aligned} a\xi\sigma^j a^{-1} =&(a_0+\cdots+a_{d-1}\sigma^{d-1})\xi\sigma^j(a'_0+\cdots+a'_{d-1}\sigma^{d-1})\\ =&(a_0\xi+a_1\varphi(\xi)\sigma+\cdots+a_{d-1}\varphi^{d-1}(\xi)\sigma^{d-1})\\ &\times(\varphi^j(a'_0)\sigma^j+\cdots+\varphi^j(a'_{d-1})\sigma^{j+d-1}), \end{aligned}$$

for each $r\in\{0,\dots,d-1\}$ we pick the term $a_r\varphi^r(\xi)\sigma^r$ in the first factor, we multiply it by the unique term $\varphi^j(a'_k)\sigma^{j+k}$ of the second factor for which $r+j+k\equiv\ell \bmod d$, and then we sum over r. We thus find that the coefficient of σ^ℓ is of the form

$$\sum_{r=0}^{d-1}\varphi^r(\xi)c_r.$$

By hypothesis, this is in $\mathbb{F}_{q^d}[1/Y]$ for each $\xi=\xi_i$, $i=0,\dots,d-1$. So if $\mathbf{c}$ is the column vector with entries $c_0,\dots,c_{d-1}$, then $Q\mathbf{c}$ has entries in $\mathbb{F}_{q^d}[1/Y]$. But $Q\in\mathrm{GL}(d,\mathbb{F}_{q^d})$, and so $\mathbf{c}$ has entries in $\mathbb{F}_{q^d}[1/Y]$. Clearly

$$c_r=\begin{cases} a_r\varphi^{r+j}(a'_{\ell-r-j}) & \text{if } r+j\le\ell,\\ Za_r\varphi^{r+j}(a'_{d+\ell-r-j}) & \text{if } \ell<r+j\le d+\ell,\\ Z^2a_r\varphi^{r+j}(a'_{2d+\ell-r-j}) & \text{if } d+\ell<r+j\le 2d+\ell. \end{cases}$$

Recall that $a_0\ne0$. We claim that $a'_1=\cdots=a'_{d-1}=0$. Fix $k\in\{1,\dots,d-1\}$. Taking $r=0$ and $\ell=j+k$ for $j=0,\dots,d-k-1$, we see that

$$a_0a'_k,a_0\varphi(a'_k),\dots,a_0\varphi^{d-k-1}(a'_k)\in\mathbb{F}_{q^d}[1/Y].$$

But a_0 and a'_k are in $\mathbb{F}_{q^d}[[Y]]$, and as $\mathbb{F}_{q^d}[[Y]]\cap\mathbb{F}_{q^d}[1/Y]=\mathbb{F}_{q^d}$, we have

$$a_0a'_k,a_0\varphi(a'_k),\dots,a_0\varphi^{d-k-1}(a'_k)\in\mathbb{F}_{q^d}. \tag{2.1}$$

Suppose that $a'_k\ne0$. Suppose at first that $k\le d-2$. Then $a_0a'_k,a_0\varphi(a'_k)\in\mathbb{F}_{q^d}$, so that $\varphi(a'_k)/a'_k\in\mathbb{F}_{q^d}$. As $a'_k\in\mathbb{F}_{q^d}[[Y]]$, this forces a'_k to equal $\theta_k\tilde a_k$ for some $\theta_k\in\mathbb{F}_{q^d}$ and $\tilde a_k\in\mathbb{F}_q[[Y]]$. Hence $a_0\tilde a_k\in\mathbb{F}_{q^d}$. Next, taking $r=0$, and $\ell=j+k-d$ for $j=d-k,\dots,d-1$, we similarly see that

$$Za_0\varphi^{d-k}(a'_k),\dots,Za_0\varphi^{d-1}(a'_k)\in\mathbb{F}_{q^d}, \tag{2.2}$$

and so $Za_0\varphi^{d-k}(\theta_k)\tilde a_k\in\mathbb{F}_{q^d}$, and hence $Za_0\tilde a_k\in\mathbb{F}_{q^d}$. But $a_0\tilde a_k\in\mathbb{F}_{q^d}$, and so we have a contradiction. When $k=d-1$, we use (2.2) to see that $a'_k=\theta_k\tilde a_k$

for some $\theta_k \in \mathbb{F}_{q^d}$ and $\tilde{a}_k \in \mathbb{F}_q[[Y]]$, and then (2.1) yields a contradiction. Thus $a'_1 = \cdots = a'_{d-1} = 0$.

Hence $a^{-1} = a'_0 \in \mathbb{F}_{q^d}(Y)$. Thus $a \in \mathbb{F}_{q^d}(Y)$, so that $a_1 = \cdots = a_{d-1} = 0$. As $\alpha \in \tilde{\Gamma}$, $a\xi_i\sigma^j a^{-1} = a_0\varphi^j(a_0^{-1})\xi_i\sigma^j$ is an $\mathbb{F}_{q^d}[1/Y]$-multiple of σ^j. But $a_0^{-1} = a'_0 \in \mathbb{F}_{q^d}[[Y]]$, and so $a_0\varphi^j(a_0^{-1})\xi_i \in \mathbb{F}_{q^d}[1/Y] \cap \mathbb{F}_{q^d}[[Y]] = \mathbb{F}_{q^d}$. As above, we have $a_0 = \theta\tilde{a}_0$, where $\tilde{a}_0 \in \mathbb{F}_q[[Y]]$ and $\theta \in \mathbb{F}_{q^d}$. As $\tilde{a}_0$ belongs to the centre of $\mathcal{A}$, α is also conjugation by θ.

THEOREM 2.6: *The group Γ acts simply transitively on the set $\mathcal{V}_\Delta$ of vertices of the $\tilde{A}_n$-building Δ associated with $\mathbb{F}_q((Y))$. Moreover, Γ is a normal subgroup of $\tilde{\Gamma}$ of index $d(q^d-1)/(q-1)$.*

Proof: To prove the first statement, we need only check that if $\theta \in \mathbb{F}_{q^d}^\times$, $j \in \{0,\ldots,d-1\}$, and if conjugation α by $\theta\sigma^j$ belongs to Γ, then $j=0$ and $\theta \in \mathbb{F}_q$ (so that α is the identity automorphism). But $\alpha \in \Gamma$ implies that $\alpha(\xi) \equiv \xi \pmod{1/Y}$ for each $\xi \in \mathbb{F}_{q^d}$. But $\alpha(\xi) - \xi = \varphi^j(\xi) - \xi$, which can be zero mod $1/Y$ only if it is zero, and if this holds for all ξ, then $j = 0$. Also, $\alpha \in \Gamma$ implies that $\alpha(\sigma) - \sigma \in \mathbb{F}_{q^d} \pmod{1/Y}$. This means that $\big(\theta\varphi(\theta^{-1}) - 1\big)\sigma \in \mathbb{F}_{q^d}$, which can only happen if $\theta = \varphi(\theta)$, that is, if $\theta \in \mathbb{F}_q$.

As Γ acts transitively on $\mathcal{V}_\Delta$, we know that $\tilde{\Gamma} = \Gamma H$, where H is the stabilizer of v_0 in $\tilde{\Gamma}$. Now conjugation by $\theta\sigma^j$ equals conjugation by $\theta'\sigma^{j'}$ if and only if $\theta = \theta' \pmod{\mathbb{F}_q^\times}$ and $j = j'$. Hence $|H| = d(q^d-1)/(q-1)$. We saw above that $\Gamma \cap H = \{\mathrm{id}\}$, and so Γ has index $d(q^d-1)/(q-1)$ in $\tilde{\Gamma}$. One can easily check that conjugation by σ or by any $\theta \in \mathbb{F}_{q^d}^\times$ normalizes Γ. Hence Γ is normal in $\tilde{\Gamma}$.

§3. The isomorphism $\Pi(\mathbb{F}_{q^d}) \to \Pi(v_0)$

Recall that the projective geometry $\Pi(v_0)$ of neighbours of v_0 in the $\tilde{A}_n$-building associated to $\mathbb{F}_q((Y))$ is isomorphic to the flag complex $\Pi(\mathbf{V})$ of a d-dimensional vector space $\mathbf{V}$ over $\mathbb{F}_q$. We may take $\mathbf{V} = \mathbb{F}_{q^d}$, and in this section, for each $U \in \Pi(\mathbb{F}_{q^d})$, we find explicit elements b_U of $\mathcal{A}$ such that $U \to b_U.v_0$ is an isomorphism $\Pi(\mathbb{F}_{q^d}) \to \Pi(v_0)$.

For $k = 1,\ldots,n$, let $\Pi_k(\mathbb{F}_{q^d})$ denote the set of k-dimensional subspaces of $\mathbb{F}_{q^d}$. Let $U \in \Pi_k(\mathbb{F}_{q^d})$, let $u_1,\ldots,u_k$ be a basis of U, and let $m_1,\ldots,m_k$ be integers. Let $[U;m_1,\ldots,m_k]$ denote the determinant of the $k\times k$ matrix whose (i,j)-th entry is $\varphi^{m_j}(u_i)$. Let $u'_1,\ldots,u'_k$ be another basis of U. Write $u'_j = \sum_{i=1}^d t_{i,j}u_i$, where $t_{i,j} \in \mathbb{F}_q$ for each i,j. Then the determinant of the $k\times k$ matrix whose (i,j)-th entry is $\varphi^{m_j}(u'_i)$ is $t[U;m_1,\ldots,m_k]$, where $t = \det(t_{i,j})$. Note that $0 \neq t \in \mathbb{F}_q$. Hence $[U;m_1,\ldots,m_k]$ is independent, modulo $\mathbb{F}_q^\times$, of the choice of the basis $u_1, \ldots, u_k$ of U. Moreover, if $[U; m_1, \ldots, m_k] \neq 0$, and if $n_1, \ldots, n_k$ are integers, then $[U; n_1, \ldots, n_k]/[U; m_1, \ldots, m_k]$, where both numerator and denominator are calculated using the same basis for U, is independent of that basis.

LEMMA 3.1: *If $U \in \Pi_k(\mathbb{F}_{q^d})$, then $[U; 0, -1, \ldots, -(k-1)] \neq 0$.*

Proof: This is obvious if $k = 1$. When $k = 2$, let u_1, u_2 be a basis for U. Then $[U; 0, -1] = u_1\varphi^{-1}(u_2) - \varphi^{-1}(u_1)u_2$. If this is 0, then $\varphi^{-1}(u_2/u_1) = u_2/u_1$, which implies that $u_2/u_1 \in \mathbb{F}_q$, which is impossible. Now let $k > 2$, and assume the result for $k-1$. Let $u_1, u_2, \ldots, u_k$ be a basis for some $U \in \Pi_k(\mathbb{F}_{q^d})$. Suppose that $[U; 0, -1, \ldots, -(k-1)] = 0$. Then $\det(A) = 0$, where A is the $k \times k$ matrix whose (i,j)-th entry is $\varphi^{-(i-1)}(u_j)$. Subtract $u_1/\varphi^{-1}(u_1)$ times row 2 from row 1, then $\varphi^{-1}(u_1)/\varphi^{-2}(u_1)$ times row 3 from row 2, etc. This results in a matrix all of whose entries in the first column are 0, except the last, $\varphi^{-(k-1)}(u_1)$. So $0 = \det(A) = (-1)^{k-1}\varphi^{-(k-1)}(u_1)\det(B)$, where B is the $(k-1)\times(k-1)$-matrix whose (i,j)-th entry is $\varphi^{-(i-1)}(u'_{j+1})$, where $u'_j = u_j - (u_1/\varphi^{-1}(u_1))\varphi^{-1}(u_j)$. Thus $\det(B) = 0$, and the induction hypothesis implies that $u'_k = a_2u'_2 + \cdots + a_{k-1}u'_{k-1}$ for some $a_2, \ldots, a_{k-1} \in \mathbb{F}_q$. Thus

$$u_1\varphi^{-1}\Big(u_k - \sum_{j=2}^{k-1} a_j u_j\Big) - \varphi^{-1}(u_1)\Big(u_k - \sum_{j=2}^{k-1} a_j u_j\Big) = 0,$$

and so by the $k = 2$ case we must have $u_k - \sum_{j=2}^{k-1} a_j u_j = a_1u_1$ for some $a_1 \in \mathbb{F}_q$. This is a contradiction.

Now let $U \in \Pi_k(\mathbb{F}_{q^d})$. By Lemma 3.1 we can form the following element b_U of $\mathcal{A}$:

$$b_U = \sum_{j=0}^{d-1} \sigma^{-j} \frac{[U; j, -1, \ldots, -(k-1)]}{[U; 0, -1, \ldots, -(k-1)]}. \tag{3.1}$$

When $U = \mathbb{F}_q u$ is one-dimensional, then b_U equals the b_u used in the last section:

$$b_U = \sum_{j=0}^{d-1} \sigma^{-j} \frac{\varphi^j(u)}{u} = u\Big(\sum_{j=0}^{d-1} \sigma^{-j}\Big)u^{-1} = ub_1u^{-1} = b_u.$$

Notice also that in (3.1), the coefficient of σ^{-j} is zero for $j = d-k+1, \ldots, d-1$.

LEMMA 3.2: *Let $U \in \Pi_k(\mathbb{F}_{q^d})$ and $W \in \Pi_{k+1}(\mathbb{F}_{q^d})$, with $U \subset W$. Then*

$$b_W = \frac{Z}{Z-1} b_U b_y \tag{3.2}$$

for (modulo $\mathbb{F}_q^\times$*)*

$$y = [W;0,-1,\ldots,-k]/[U;-1,\ldots,-k]. \tag{3.3}$$

Proof: For any $y \in \mathbb{F}_q^\times$, we have $b_y = yb_1y^{-1}$, and so

$$b_y^{-1} = yb_1^{-1}y^{-1} = \frac{Z}{Z-1}y(1-\sigma^{-1})y^{-1} = \frac{Z}{Z-1}\Big(1-\sigma^{-1}\frac{\varphi(y)}{y}\Big).$$

So we need only show that for y as in (3.3),

$$b_W\Big(1-\sigma^{-1}\frac{\varphi(y)}{y}\Big) = b_U.$$

The left hand side equals

$$\sum_{j=0}^{d-1}\sigma^{-j}\frac{[W;j,-1,\ldots,-k]}{[W;0,-1,\ldots,-k]} - \sum_{j=0}^{d-1}\sigma^{-(j+1)}\frac{[W;j+1,0,\ldots,-(k-1)]}{[W;1,0,\ldots,-(k-1)]}\frac{\varphi(y)}{y} =$$
$$\sum_{j=0}^{d-1}\sigma^{-j}\frac{[W;j,-1,\ldots,-k]}{[W;0,-1,\ldots,-k]} - \sum_{j=1}^{d}\sigma^{-j}\frac{[W;j,0,\ldots,-(k-1)][U;-1,\ldots,-k]}{[W;0,-1,\ldots,-k][U;0,-1,\ldots,-(k-1)]}.$$

Notice that the $j=d$ term in the last sum is 0, and that the $j=0$ term in the second last sum is 1. For $1 \le j \le d-1$, collecting the terms in σ^{-j} for these two sums, we see that the result will be proved once we check the identity

$$\begin{aligned} &[W;j,-1,\ldots,-k][U;0,-1,\ldots,-(k-1)] \\ &-[W;j,0,\ldots,-(k-1)][U;-1,-2,\ldots,-k] \\ &=[W;0,-1,\ldots,-k][U;j,-1,\ldots,-(k-1)]. \end{aligned} \tag{3.4}$$

Let $w_1,\ldots,w_{k+1}$ be a basis of W such that $w_1,\ldots,w_k$ is a basis of U. The desired identity is of the form $ab-cd=ef$, where $a,\ldots,f$ are certain determinants. Now a, c and f are determinants of matrices whose first columns have entries $\varphi^j(w_1),\ldots,\varphi^j(w_{k+1})$ (the last of these not used in f). If we replace these columns by a column vector $\mathbf{x} = (x_1,\ldots,x_{k+1})^t$, we see that $ab-cd-ef$ is linear in $\mathbf{x}$. When we set $\mathbf{x} = \mathbf{x}_\nu = (\varphi^{-\nu}(w_1),\ldots,\varphi^{-\nu}(w_{k+1}))^t$, where $0 \le \nu \le k$, then a moment's thought shows that $ab-cd-ef=0$. But Lemma 3.1 shows that the $k+1$ vectors $\mathbf{x}_\nu$ are linearly independent. Thus $ab-cd-ef$ equals 0 for any $\mathbf{x}$, and so, in particular, for $\mathbf{x} = (\varphi^j(w_1),\ldots,\varphi^j(w_{k+1}))^t$.

LEMMA 3.3: *If $U \in \Pi_k(\mathbb{F}_{q^d})$, then*

$$b_U^{-1} = \frac{Z}{Z-1}\sum_{j=0}^{k}(-1)^j \frac{[U;0,-1,\ldots,\widehat{-j},\ldots,-k]}{[U;-1,-2\ldots,-k]}\sigma^{-j}, \tag{3.5}$$

where $\widehat{-j}$ indicates that the index $-j$ is omitted.

Proof: Let $k=1$, and let $U = \mathbb{F}_q u$. Then $b_u = ub_1u^{-1}$, so that

$$\begin{aligned} b_u^{-1} = ub_1^{-1}u^{-1} = u\frac{Z}{Z-1}(1-\sigma^{-1})u^{-1} &= \frac{Z}{Z-1}\Big(1 - \frac{u}{\varphi^{-1}(u)}\sigma^{-1}\Big) \\ &= \frac{Z}{Z-1}\Big(1 - \frac{[U;0]}{[U;-1]}\sigma^{-1}\Big). \end{aligned}$$

Suppose now that the formula has been proved for all k-dimensional subspaces of $\mathbb{F}_{q^d}$, and let $W \in \Pi_{k+1}(\mathbb{F}_{q^d})$. Choose any k-dimensional subspace U of W. Let $w_1,\ldots,w_{k+1}$ be a basis of W such that $w_1,\ldots,w_k$ is a basis of U. By Lemma 3.2, we have

$$b_W = \frac{Z}{Z-1}b_U b_y \quad \text{for } y = [W;0,-1,\ldots,-k]/[U;-1,\ldots,-k].$$

Using the induction hypothesis, we have, writing $(-1)^jC_j$ for the coefficient of σ^{-j} in (3.5),

$$b_W^{-1} = \frac{Z-1}{Z}b_y^{-1}b_U^{-1} = \frac{Z}{Z-1}\Big(1 - \frac{y}{\varphi^{-1}(y)}\sigma^{-1}\Big)\Big(\sum_{j=0}^{k}(-1)^jC_j\sigma^{-j}\Big).$$

Multiplying out the second and third factors, we see that the coefficient of $\sigma^{-(k+1)}$ is

$$(-1)^{k+1}\frac{y}{\varphi^{-1}(y)}\varphi^{-1}(C_k) = (-1)^{k+1}\frac{[W;0,-1,\ldots,-k]}{[W;-1,-2,\ldots,-(k+1)]},$$

as desired. For $1 \le j \le k$, the coefficient of σ^{-j} is

$$(-1)^jC_j - (-1)^{j-1}\frac{y}{\varphi^{-1}(y)}\varphi^{-1}(C_{j-1}).$$

After a little algebra, we see that this equals the desired value $(-1)^j[W;0,-1,\ldots,\widehat{-j},\ldots,-(k+1)]/[W;-1,-2\ldots,-(k+1)]$ if and only if

$$\begin{aligned} &[U;0,-1,\ldots,\widehat{-j},\ldots,-k][W;-1,-2\ldots,-(k+1)] \\ &\qquad + [U;-1,-2,\ldots,\widehat{-j},\ldots,-(k+1)][W;0,-1\ldots,-k] \\ &\qquad = [U;-1,-2,\ldots,-k][W;0,-1,\ldots,\widehat{-j},\ldots,-(k+1)]. \end{aligned}$$

The proof of this identity is very similar to the proof of (3.4), and we omit it.

PROPOSITION 3.4: *For each $U \in \Pi(\mathbb{F}_{q^d})$, conjugation by b_U is an element of Γ. Let $g_U \in \mathrm{GL}(d, \mathbb{F}_q[[Y]])$ denote the image of b_U under the embedding (1.2). Let v_0 be the standard base vertex of the $\tilde{A}_n$ building Δ associated with $\mathbb{F}_q((Y))$. Then for each $k \in \{1, \dots, n\}$, the type k neighbours of v_0 in Δ are precisely the vertices $g_U v_0$, where $U \in \Pi_k(\mathbb{F}_{q^d})$. The correspondence $U \leftrightarrow g_U v_0$ is an isomorphism of the projective geometries $\Pi(\mathbb{F}_{q^d})$ and $\Pi(v_0)$.*

Proof: The first statement is clear from Lemma 2.1, Lemma 3.2 and induction. It is clear from the definition of b_U that g_U has entries in $Z^{-1}\mathbb{F}_q[X] \subset \mathbb{F}_q[[Y]]$, and so $g_U L_0 \subset L_0$. Lemma 3.3 shows that $Y g_U^{-1} = (Z-1) g_U^{-1}$ has entries in $\mathbb{F}_q[X] \subset \mathbb{F}_q[[Y]]$, and so $Y L_0 \subset g_U L_0$. We have seen that the norm $N(b_1)$ equals $((Z-1)/Z)^{d-1}$. A simple induction based on Lemma 3.2 implies that $N(b_U) = ((Z-1)/Z)^{d-k}$ if U is k-dimensional. Thus $\det(g_U) = N(b_U)$ has valuation $d-k$. Thus $g_U v_0$ is a neighbour of v_0 of type k. Let $U \in \Pi_i(\mathbb{F}_{q^d})$ and $V \in \Pi_j(\mathbb{F}_{q^d})$, with $U \subset V$. If $j = i+1$, then $g_U v_0$ and $g_V v_0$ are incident in $\Pi(v_0)$ by Lemma 3.2. For we can write

$$b_V = \frac{Z}{Z-1} b_U b_y \quad \text{for some } y \in \mathbb{F}_{q^d}^{\times}.$$

Thus

$$d(g_V v_0, g_U v_0) = d(g_U g_y v_0, g_U v_0) = d(g_y v_0, v_0) = 1,$$

by Lemma 2.2. For general $j > i$, choose any subspaces $U_i \subset U_{i+1} \subset \cdots \subset U_j$ such that $U_i = U$, $U_j = V$, and $\dim(U_\nu) = \nu$ for each ν. By the above, $g_{U_\nu} v_0$ and $g_{U_{\nu+1}} v_0$ are incident in $\Pi(v_0)$ for $\nu = i, \dots, j-1$. As $\Pi(v_0)$ is a projective geometry and each $g_{U_\nu} v_0$ has type ν, this implies that $g_U v_0$ and $g_V v_0$ are incident.

Finally, we show that the map $U \mapsto g_U v_0$ is a bijection $\Pi(\mathbb{F}_{q^d}) \to \Pi(v_0)$ by showing that it is injective. We know from Lemma 2.2 that it is a bijection between the 1-dimensional subspaces of $\mathbb{F}_{q^d}$ and the type 1 neighbours of v_0. Suppose that $V, V' \in \Pi_k(\mathbb{F}_{q^d})$ are distinct, where $k > 1$. Then V contains exactly $N = (q^k - 1)/(q-1)$ 1-dimensional subspaces, U_j, $j = 1, \dots, N$, say. Choose a one dimensional subspace U of V' which is not contained in V. If $g_V v_0 = g_{V'} v_0$ were to hold, then $g_V v_0$ would be incident in $\Pi(v_0)$ with the $N+1$ type 1 vertices $g_{U_j} v_0$, $j = 1, \dots, N$, and $g_U v_0$, which is impossible. Thus $g_V v_0 \neq g_{V'} v_0$.

By Proposition 2.2 in [2], we know that the image of Γ in $\mathrm{PGL}(d, \mathbb{F}_q((Y)))$ has a presentation with generators the g_U's, $U \in \Pi(\mathbb{F}_{q^d})$, and relations (1) $g_U g_V g_W = 1$ for certain $U, V, W \in \Pi(\mathbb{F}_{q^d})$ satisfying $\dim(U) + \dim(V) + \dim(W) = d$ (see [1, §2]), and (2) $g_U g_{\lambda(U)} = 1$ for each $U \in \Pi(\mathbb{F}_{q^d})$, where λ is an involution of $\Pi(\mathbb{F}_{q^d})$ satisfying $\dim(\lambda(U)) = d - \dim(U)$ for each U (see [1, Lemma 2.1]). It is routine to rephrase these statements in terms of the b_U's:

COROLLARY 3.5: *Suppose that U and W are subspaces of $\mathbb{F}_{q^d}$, with $U \subsetneqq W$. Then there is a subspace V of $\mathbb{F}_{q^d}$ such that* $\dim(W) = \dim(U) + \dim(V)$ *and*

$$b_W = \frac{Z}{Z-1} b_U b_V. \tag{3.6}$$

Moreover, there is an involution λ of $\Pi(\mathbb{F}_{q^d})$ such that $\dim(\lambda(U)) = d - \dim(U)$ *for each $U \in \Pi(\mathbb{F}_{q^d})$, and such that*

$$b_{\lambda(U)} = \frac{Z-1}{Z} b_U^{-1}. \tag{3.7}$$

We omit the proof of Corollary 3.5. In (3.6), we have in general no completely explicit "formula" for V in terms of U and W when $\dim(W/U) > 1$. However, a routine induction on $\dim(W/U)$ shows that, modulo $\mathbb{F}_q^\times$,

$$[V; 0, -1, \ldots, -(j-1)] = \frac{[W; 0, -1, \ldots, -(k-1)]}{[U; -j, \ldots, -(k-1)]}.$$

In (3.7), we have no closed formula for $\lambda(U)$, except when $\dim(U) \leq 2$. If $U = \mathbb{F}_q u$ is 1-dimensional, then one can easily show that

$$\lambda(U) = \{x \in \mathbb{F}_{q^d} : \mathrm{Tr}(x/u) = 0\} = u\{1\}^\perp,$$

where for $S \subset \mathbb{F}_{q^d}$, $S^\perp = \{x \in \mathbb{F}_{q^d} : \mathrm{Tr}(sx) = 0 \text{ for all } s \in S\}$. If $U \in \Pi_2(\mathbb{F}_{q^d})$, then one can show that

$$\lambda(U) = [U; 0, -1] \varphi^{-1}(U^\perp).$$

In general, there is no simple relationship between $\lambda(U)$ and $U^\perp$. However, (3.7) implies various relations between some of the numbers $[U; m_1, \ldots, m_k]$ and $[\lambda(U); n_1, \ldots, n_{d-k}]$. If we also use the relation

$$[U^\perp; -k, \ldots, -(d-1)]\theta = [U; 0, -1, \ldots, -(k-1)] \qquad \text{modulo } \mathbb{F}_q^\times,$$

where $\theta = [\mathbb{F}_{q^d}; 0, -1, \ldots, -(d-1)]$, which is a consequence of [6, p. 166], we can show that, for any $U \in \Pi_k(\mathbb{F}_{q^d})$,

$$[\lambda(U)^\perp; 0, -1, \ldots, -(k-1)] = 1/[U; 0, -1, \ldots, -(k-1)] \qquad \text{modulo } \mathbb{F}_q^\times.$$

Remark 3.6: Recall that the algebra $\mathcal{A}$ was defined using an automorphism φ of $\mathbb{F}_{q^d}$ which generates the Galois group of $\mathbb{F}_{q^d}$ over $\mathbb{F}_q$, and is therefore of the form $x \mapsto x^{q^k}$, where $1 \leq k \leq d$ and g.c.d.$(k, d) = 1$. We therefore have $\phi(n)$ different algebras $\mathcal{A}$, and so $\phi(n)$ different groups Γ, where ϕ denotes Euler's ϕ-function. Note that if $\mathcal{A} = \mathbb{F}_{q^d}(Z)[\sigma]$ and $\mathcal{A}' = \mathbb{F}_{q^d}(Z')[\sigma']$, where $\sigma x \sigma^{-1} = \varphi(x)$ and $\sigma^d = Z$, while $\sigma' x (\sigma')^{-1} = \varphi^{-1}(x)$ and $(\sigma')^d = Z'$, then there is an isomorphism $\mathcal{A} \to \mathcal{A}'$ mapping Z to $1/Z'$, σ to $(\sigma')^{-1}$, and x to x for each $x \in \mathbb{F}_{q^d}$. This halves the number of different groups Γ.

In [1], the first author worked with a single algebra $\mathcal{A}_0 = \mathbb{F}_{q^d}(Z_0)[\sigma_0]$ where $\sigma_0 x \sigma_0^{-1} = \varphi_0(x) = x^q$ for $x \in \mathbb{F}_{q^d}$, and where $\sigma_0^d = Z_0$. Notice that there is an algebra homomorphism $T : \mathcal{A} \to \mathcal{A}_0$ mapping σ to σ_0^k (where $\varphi(x) = x^{q^k}$, as above), Z to Z_0^k, and x to x for $x \in \mathbb{F}_{q^d}$. If $d = 4$ (so that $\phi(d)/2 = 1$) and $k = 1$, then Z times the elements b_U defined in (3.1) are just the b_U's defined in [1, §4]. If $d = 5$ (so that $\phi(d)/2 = 2$) and $k = 2$, then the image under T of Z times the elements b_U defined in (3.1) are just the b_U's defined in [1, §4]. If $d = 5$ and $k = 1$, elements of $\mathcal{A}_0 = \mathcal{A}$ corresponding to the b_U's were not found in [1].

Let $\Gamma \leq \mathrm{Aut}(\mathcal{A})$ be as above. If $\beta \in \mathbb{F}_{q^d}^\times$ satisfies $\beta^d \in \mathbb{F}_q$, then Lemma 3.1 of [1] gives a way of constructing another $\tilde{A}_n$-group Γ^β from Γ. Using Proposition 2.6 of [1], one may easily realize Γ^β as a subgroup of $\tilde{\Gamma}$; one uses the elements $b'_U = b_U \beta^{-\dim(U)}$ of $\mathcal{A}$, and the fact that $t b_U t^{-1} = b_{tU}$ for $t \in \mathbb{F}_{q^d}^\times$.

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